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A general and practical method for calculating cosmological distances

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A. Symmetry: The relation between D_{xy} and D_{yx}

The proof in this appendix follows closely the proof presented in Kayser (1985). For completeness we note that after the original derivation by Kayser (1985) an equivalent equation was derived by Linder (1988). We rewrite the differential equation, Eq. (33), for the angular size distance in the normal form:

$$a_2 D''(z) + a_1(z)\, D'(z) + a_0(z)\, D(z) = 0 \tag{A1}$$

with the coefficient functions

$$a_2(z) = Q(z) \tag{A2}$$

$$a_1(z) = \frac{2Q(z)}{1+z} + \frac{1}{2}\, Q'(z) \tag{A3}$$

$$a_0(z) = \frac{3}{2}\, \eta\, \Omega_0 (1+z) \quad . \tag{A4}$$

Now let $D^{(1)}$ and $D^{(2)}$ be two solutions of Eq. (A1) which build a fundamental system, i.e. the Wronskian for these two solutions does not vanish:

$$W(z) = \begin{vmatrix} D^{(1)} & D^{(2)} \\ \frac{dD^{(1)}}{dz} & \frac{dD^{(2)}}{dz} \end{vmatrix} \neq 0 \quad \forall z \quad . \tag{A5}$$

Every solution D_i of Eq. (A1) can then be written as a linear combination of $D^{(1)}$ and $D^{(2)}$:

$$D_i = \alpha_i\, D^{(1)} + \beta_i\, D^{(2)}; \quad with\ \alpha_i,\, \beta_i = const \quad . \tag{A6}$$

The angular size distances are special solutions D_x of Eq. (A1) fulfilling the following boundary conditions:

$$D_x = 0 \quad for \quad z = z_x \tag{A7}$$

and

$$\frac{dD_x}{dz} = b(z_x) \quad for \quad z = z_x \tag{A8}$$

with

$$b(z_x) = \frac{c}{H_0} \left((1+z_x)\sqrt{Q(z_x)} \right)^{-1} sign(t_y - t_x) \quad , \tag{A9}$$

compare Eq. (36). From Eq. (A6) we obtain

$$0 = \alpha_i\, D^{(1)}(z_x) + \beta_i\, D^{(2)}(z_x) \tag{A10}$$

and

$$b(z_x) = \alpha_i \left. \frac{dD^{(1)}}{dz} \right|_{z=z_x} + \beta_i \left. \frac{dD^{(2)}}{dz} \right|_{z=z_x} \quad . \tag{A11}$$

These equations can easily be solved for α_i and β_i:

$$\alpha_i = \beta_i\, \frac{D^{(2)}(z_x)}{D^{(1)}(z_x)} \tag{A12}$$

$$\beta_i = \frac{b(z_x)\, D^{(1)}(z_x)}{W(z_x)} \tag{A13}$$

and inserting α_i and β_i back into Eq. (A6) we obtain for the special solutions D_x:

$$D_x(z) = \frac{b(z_x)}{W(z_x)} \Big(D^{(1)}(z_x)\, D^{(2)}(z) - D^{(1)}(z)\, D^{(2)}(z_x) \Big) \quad . \tag{A14}$$

If we now consider a second special solution D_y we find the relation

$$\frac{D_x(z_y)}{D_y(z_x)} = -\frac{b(z_x)}{b(z_y)}\, \frac{W(z_y)}{W(z_x)} \quad . \tag{A15}$$

The Wronskians can be calculated using Liouville's formula:

$$W(z) = W(z_0)\, \exp \int_z^{z_0} a_2(z)\, dz \quad , \tag{A16}$$

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where z_0 is arbitrary. Thus

$$\frac{D_x(z_y)}{D_y(z_x)} = -\frac{b(z_x)}{b(z_y)} \exp \int_{z_y}^{z_x} \frac{a_1(z)}{a_2(z)}\, dz \tag{A17}$$

and after inserting a_0, a_1 and a_2 from Eqs. (A2), (A3) and (A4) as well as $b(z_x)$ and $b(z_y)$ from Eq. (A9) and integration we finally obtain for the angular size distances (cf. Eq. (37)) the relation

$$\frac{D_{xy}}{D_{yx}} = \frac{1+z_x}{1+z_y} \quad . \tag{A18}$$

B. Special cases

For certain special cases the differential equation can be simplified and sometimes analytically solved.

B.1. $\Omega_0 = 0$

A glance at Eq. (33) shows that for $\Omega_0 = 0$ the third term on the left hand side of Eq. (33) vanishes; one thus has a first order differential equation for D'. (Of course η has no meaning for $\Omega_0 = 0$.) Due to the fact that a vanishing Ω_0 also simplifies $Q(z)$, it is possible to calculate the angular size distance analytically. Since in this case the angular size distance is determined exclusively by global effects, one can use an approach based on global geometry.[1] Depending on the value of λ_0, one can use the following expression to calculate $\chi_{xy} = \chi(y) - \chi(x)$ (Feige 1992)

$$\chi(z) = \begin{cases} arccosh(\psi) & for \quad \lambda_0 < 0 \\ \ln(1+z) & for \quad \lambda_0 = 0 \\ arcsinh(\psi) & for \quad 0 < \lambda_0 < 1 \\ z & for \quad \lambda_0 = 1 \\ \arcsin(\psi) & for \quad \lambda_0 > 1 \end{cases} , \tag{B1}$$

where $\psi := (1+z)\sqrt{\frac{|1-\lambda_0|}{|\lambda_0|}}$. The relationship between χ and the angular size distance D is

$$D_{xy} = \frac{R_0}{(1+z_y)} \begin{cases} \sinh\chi & for\ k = -1 \\ \chi & for\ k = 0 \\ \sin\chi & for\ k = +1 \end{cases} , \tag{B2}$$

as discussed below in Sect B.3.

B.2. $\eta = 0$

In the case $\eta = 0$ the third term on the left hand side of Eq. (33) vanishes; one thus has a first order differential equation for D'. Assuming $D' \neq 0$, Eq. (33) can be written as

$$\frac{D''}{D'} = -\frac{2}{1+z} - \frac{1}{2}\frac{Q'(z)}{Q(z)} \quad . \tag{B3}$$

This equation can be solved in two steps. For D' we obtain

$$D' = \frac{c_1}{\sqrt{Q(z)}\,(1+z)^2} \tag{B4}$$

and consequently for D

$$D = \int \frac{c_1}{\sqrt{Q(z)}\,(1+z)^2} + c_2 \quad . \tag{B5}$$

The constants c_1, c_2 are determined by the appropriate boundary conditions (Eqs. (34) and (35)). We then find the solution (see also Schneider et al. (1992) – hereafter SEF – for an equivalent discussion with $\lambda_0 = 0$)

$$D_{xy} = \frac{c}{H_0}(1+z_x)(\omega(z_y) - \omega(z_x)), \tag{B6}$$

where

$$\omega(z) = \int_0^z \frac{dz'}{(1+z')^2\sqrt{(1+z')^2(\Omega_0 z' + 1 - \lambda_0) + \lambda_0}}, \tag{B7}$$

or, perhaps more convenient,

$$D_{xy} = \frac{c}{H_0}(1+z_x)\int_{z_x}^{z_y} \frac{dz}{(1+z)^2\sqrt{Q(z)}} \quad . \tag{B8}$$

For $\lambda_0 = 0$ there is an analytic solution (see Sect. B.4).

B.3. $\eta = 1$

The case $\eta = 1$ has all matter distributed homogeneously. Due to homogeneity, the matter *locally* affecting the propagation of light is known when the *global* geometry is known, so that the 'classical' approach of relating global geometry to observable relations is a better approach than using (the simplified form of) Eq. (33). This approach offers an analytic solution. Here, we simply sketch the most important points; the interested reader can refer to Feige (1992) for a good description of this method.

The angular size distance in this case is

$$D_{xy} = R_y\sigma_{xy} = \frac{R_0\sigma_{xy}}{(1+z_y)}, \tag{B9}$$

where σ is the radial coordinate in the Robertson-Walker metric (cf. Eq. (1)) and thus

$$D_{yx} = \frac{R_0\sigma_{xy}}{1+z_x} = D_{xy}\left(\frac{1+z_y}{1+z_x}\right), \tag{B10}$$

[1] See the discussion in Sect. B.3.

since this angle is inversely proportional to R for constant σ and physical size. (The value of R at the time the light rays defining the angle intersect is important.)

Since σ is given by

$$\sigma = F(\chi) = \begin{cases} \sinh\chi & for\ k\ =\ -1 \\ \chi & for\ k\ =\ 0 \\ \sin\chi & for\ k\ =\ +1 \end{cases}, \tag{B11}$$

an expression for $\chi(z)$ is sufficient for calculating the angular size distance D (and of course the luminosity distance D^L (via Eq. (19)) and the 'coordinate distance' σ (via Eq. (B11)). In general, $\sigma_{xy} \neq \sigma_y - \sigma_x$; however, $\chi_{xy} = \chi_y - \chi_x$, so that

$$\sigma_{xy} = F(\chi_{xy}) \tag{B12}$$

where F is given by Eq. (B11). Using Eq. (23) one can calculate

$$\chi_{xy} = \frac{D^P}{R_0} = \frac{c}{H_0 R_0}\int\limits_{z_x}^{z_y} \frac{dz}{\sqrt{Q(z)}}. \tag{B13}$$

In the general case, Eq. (23) can be solved by elliptic integrals, as explained in Feige (1992). For the cases $\lambda_0 = 0$ and $\Omega_0 = 0$ the formulae using elliptic integrals break down; in these cases, easier analytic formulae, which fortunately exist, can be used. The case $\Omega_0 = 0$ has been discussed above. The case $\lambda_0 = 0$ will be discussed below. *Again, we stress that the differential equation derived in Sect. 4 is completely general and can be used in* all *cases.*

B.4. $\lambda_0 = 0$

For $\lambda_0 = 0$, there is in general no simpler solution. This case has been discussed by Dyer and Roeder for $\eta = 0$ (1972) and for general η values (1973). They point out the interesting result that the maximum in the angular size distance from $z_1 = 0$ to z_2 increases monotonically from 1.25 to ∞ as η decreases from 1 to 0. See also the discussion (with a differing notation!) in Sect. 4.5.3 in SEF. However, some solutions exist for special values of Ω_0 and η. The case $\Omega_0 = 0$ has been discussed in Sect. B.1 above; the value of η is of course irrelevant in this case. With the exception of Sect B.4.3 below, in the following we simply quote results from SEF in our notation.

B.4.1. $\lambda_0 = 0$ and $\eta = 0$

As discussed above, for $\eta = 0$ Eq. (33) is effectively a first order equation for D'. For $\lambda_0 = 0$ $Q(z)$ is sufficiently simplified to allow an analytic solution. Recalling Eq. (B6),

$$D_{xy} = \frac{c}{H_0}(1+z_x)(\omega(z_y) - \omega(z_x)),$$

Eq. (B7) simplifies to

$$\omega(z) = \int\limits_0^z \frac{dz'}{(1+z')^3\sqrt{\Omega_0 z' + 1}}, \tag{B14}$$

which has the solution:

$$\omega(z) = \begin{cases} \frac{3\Omega_0^2}{4(\Omega_0-1)^{\frac{5}{2}}}\arctan(\psi) + \frac{3}{4(\Omega_0-1)^2} \times \\ \left(\frac{\left(\Omega_0 z + \frac{5\Omega_0}{3} - \frac{2}{3}\right)\sqrt{\Omega_0 z+1}}{(1+z)^2} - \frac{5\Omega_0}{3} + \frac{2}{3}\right) & (I) \\ \frac{2}{5}\left(1 - (\Omega_0 z + 1)^{-\frac{5}{2}}\right) & (II) \\ \frac{3\Omega_0^2}{4(1-\Omega_0)^{\frac{5}{2}}}\, arctanh(\psi) + \frac{3}{4(1-\Omega_0)^2} \times \\ \left(\frac{\left(\Omega_0 z + \frac{5\Omega_0}{3} - \frac{2}{3}\right)\sqrt{\Omega_0 z+1}}{(1+z)^2} - \frac{5\Omega_0}{3} + \frac{2}{3}\right) & (III) \end{cases} \tag{B15}$$

with

$$\psi = \begin{cases} \left(\frac{\sqrt{\Omega_0-1}(1+\sqrt{\Omega_0 z+1})}{\Omega_0 - 1 + \sqrt{\Omega_0 z+1}}\right) & (I) \\ \left(\frac{\sqrt{1-\Omega_0}(1+\sqrt{\Omega_0 z+1})}{\Omega_0 - 1 + \sqrt{\Omega_0 z+1}}\right) & (III) \end{cases} \tag{B16}$$

and

$$\begin{array}{ll} \text{case I:} & \Omega_0 > 1 \\ \text{case II:} & \Omega_0 = 1\,. \\ \text{case III:} & 0 < \Omega_0 < 1 \end{array} \tag{B17}$$

Note that in SEF, the text at the top of page 137 is unclear—the expression in parentheses in the denominator of the first term $(\Omega - 1)$ for the $\Omega > 1$ case has to be replaced with $(1 - \Omega)$ as well for $\Omega < 1$. Note also that $\Omega \equiv \Omega_0$ and that after page 131 $\lambda_0 = 0$ is always assumed.

B.4.2. $\lambda_0 = 0$ and $\Omega_0 = 1$

For $\Omega_0 = 1$ and $\lambda_0 = 0$ (the Einstein-de Sitter model) we have the solution

$$D_{xy} = \frac{c}{H_0}\frac{1}{2\beta}\left(\frac{(1+z_y)^{\beta-\frac{5}{4}}}{(1+z_x)^{\beta+\frac{1}{4}}} - \frac{(1+z_x)^{\beta-\frac{1}{4}}}{(1+z_y)^{\beta+\frac{5}{4}}}\right), \tag{B18}$$

where

$$\beta := \frac{1}{4}\sqrt{25 - 24\eta}. \tag{B19}$$

B.4.3. $\lambda_0 = 0$ and $\eta = 1$

For $\eta = 1$ the special case of the expression for $\chi(z)$ for $\lambda_0 = 0$ is (Feige 1992)

$$\chi(z) = -2\begin{cases} \arcsin\left(\sqrt{\frac{\Omega_0 - 1}{\Omega_0(1+z)}}\right) & (\Omega_0 > 1) \\ \sqrt{\frac{1}{1+z}} & (\Omega_0 = 1) \\ arcsinh\left(\sqrt{\frac{1-\Omega_0}{\Omega_0(1+z)}}\right) & (0 < \Omega_0 < 1) \end{cases}, \tag{B20}$$

where $\chi_{xy} = \chi(y) - \chi(x)$. (It is obvious that in the case $\lambda_0 = \Omega_0 = 0$ Eq. (B1) should be used.) From this, it is possible to obtain a general expression for the angular size distance (see, e.g., SEF):

$$D_{xy} = \frac{c}{H_0}\frac{2}{\Omega_0^2}(1+z_x)\left(R_1(z_y)R_2(z_x) - R_1(z_x)R_2(z_y)\right), \quad \text{(B21)}$$

with

$$R_1(z) = \frac{\Omega_0 z - \Omega_0 + 2}{(1+z)^2} \quad \text{(B22)}$$

and

$$R_2(z) = \frac{\sqrt{\Omega_0 z + 1}}{(1+z)^2}. \quad \text{(B23)}$$

For $z_x = 0$ and $z_y = z$ one gets for the angular size distance

$$D(z) = \frac{c}{H_0}\frac{2}{\Omega_0^2(1+z)^2} \times \left(\Omega_0 z - (2-\Omega_0)\left(\sqrt{\Omega_0 z + 1} - 1\right)\right). \quad \text{(B24)}$$

valid for $\Omega_0 > 0$. For $\Omega_0 = 0$ one obtains

$$D = \frac{c}{H_0}\frac{z\left(1+\frac{z}{2}\right)}{(1+z)^2} \quad \text{(B25)}$$

(Multiplying Eq. (B24) or Eq. (B25) with $(1+z)/R_0$ results in the respective expression for σ as a function of redshift as first derived by Mattig (1958). See also Sandage (1995), Sect. 1.6.3). In this case, the volume element given by Eq. (C4) reduces to

$$dV = 16\pi R_0^3 \frac{(\Omega_0 z - (2-\Omega_0)(\sqrt{\Omega_0 z+1} - 1)^2)}{\Omega_0^4(1+z)^3\sqrt{\Omega_0 z + 1}} \quad \text{(B26)}$$

Of course, for the physical, as opposed to comoving, density, an *additional* factor of $(1+z)^3$ must be added to the denominator.

B.4.4. $\lambda_0 = 0$ and $\eta = \frac{2}{3}$

For $\eta = \frac{2}{3}$ and $\lambda_0 = 0$ there is also an analytic solution (see SEF):

$$D_{xy} = \frac{c}{H_0}\frac{2}{3\Omega_0^2}(1+z_x) \times \left(R_1(z_x)R_2(z_y) - R_2(z_x)R_1(z_y)\right), \quad \text{(B27)}$$

with

$$R_1(z) = \frac{1}{(1+z)^2} \quad \text{(B28)}$$

and

$$R_2(z) = \frac{\sqrt{\Omega_0 z + 1}(\Omega_0 z + 3\Omega_0 - 2)}{(1+z)^2}. \quad \text{(B29)}$$

B.5. *Other cases*

We can offer no proof that no other easier solutions, either reducing Eq. (33) to a more easily (numerically) integrated form or even to an analytic solution, exist. This is left as an exercise to the interested reader. The authors are of course interested in such solutions and are willing to verify them. As far as we know, Eq. (33) must be used except in the special cases mentioned in this appendix.

B.6. *Light travel time*

Feige (1992) not only gives the distance but also the light travel time by means of elliptic integrals. As for the distance, and for the same reasons, simple analytic formulae can and must be used for the special cases $\Omega_0 = 0$ and $\lambda_0 = 0$. For $k = 0$, an analytic expression for the light travel time exists, although the elliptic integrals can also be used in this case. For completeness, we give these special cases here for the light travel time $t_{xy} = t_x - t_y$.

For $\Omega_0 = 0$ we have:

$$t(z) = \frac{1}{H_0\sqrt{|\lambda_0|}}\begin{cases} \arcsin(\psi) & \lambda_0 < 0 \\ \frac{\sqrt{|\lambda_0|}}{(1+z)} & \lambda_0 = 0 \\ arcsinh(\psi) & 0 < \lambda_0 < 1 \\ -\sqrt{|\lambda_0|}\ln(1+z) & \lambda_0 = 1 \\ arccosh(\psi) & \lambda_0 > 1 \end{cases}, \quad \text{(B30)}$$

where $\psi := \frac{1}{(1+z)}\sqrt{\frac{|\lambda_0|}{|1-\lambda_0|}}$

For $\lambda_0 = 0$ we have:

$$t(z) = A \times \begin{cases} \frac{\sqrt{(\Omega_0 z+1)(\Omega_0 - 1)}}{\Omega_0(1+z)} - \arcsin\left(\sqrt{\frac{\Omega_0 - 1}{\Omega_0(1+z)}}\right) & \Omega_0 > 1 \\ -\frac{\sqrt{\Omega_0 - 1}^3}{\Omega_0}\frac{2}{3}\left(\sqrt{\frac{1}{1+z}}\right)^3 & \Omega_0 = 1 \\ arcsinh\left(\sqrt{\frac{1-\Omega_0}{\Omega_0(1+z)}}\right) - \frac{\sqrt{(\Omega_0 z+1)(1-\Omega_0)}}{\Omega_0(1+z)} & 0 < \Omega_0 < 1 \end{cases}, \quad \text{(B31)}$$

where

$$A = -\frac{\Omega_0}{H_0(\sqrt{|\Omega_0 - 1|})^3}$$

(For $\Omega_0 = 0$ the appropriate case from Eq. (B30) must be used.)

For $k = 0$ we have:

$$t(z) = \frac{2}{3H_0} \times \begin{cases} \frac{1}{\sqrt{\Omega_0 - 1}} \arcsin(\psi) & \Omega_0 > 1 \\ \left(\sqrt{\frac{1}{1+z}}\right)^3 & \Omega_0 = 1 \\ \frac{1}{\sqrt{1-\Omega_0}} arcsinh(\psi) & 0 < \Omega_0 < 1 \end{cases}, \quad \text{(B32)}$$

where

$$\psi = \begin{cases} \sqrt{\frac{\Omega_0 - 1}{\Omega_0 (1+z)^3}} & \Omega_0 > 1 \\ \sqrt{\frac{1-\Omega_0}{\Omega_0(1+z)^3}} & 0 < \Omega_0 < 1 \end{cases}. \quad \text{(B33)}$$

(For $\Omega_0 = 0$ the appropriate case from Eq. (B30) must be used.)

C. Volume element

Sometimes the distance is only a means of calculating the volume element at a given redshift. In the static Euclidean case the volume element is of course

$$dV = 4\pi r^2 dr. \quad \text{(C1)}$$

In the cosmological case, the volume element is, with $r = R_0\sigma$,

$$dV = 4\pi r^2 dD^P = 4\pi r^2 \frac{c}{H_0} \frac{dz}{\sqrt{Q(z)}}. \quad \text{(C2)}$$

$R_0\sigma_y$ is, for $\eta = 1$, simply $D_{y0} = (1+y)D_{0y}$; see Sect B.3. Thus, the distance D_{y0} is all that is needed to calculate the volume; this first can be calculated by Eq. (33) with $\eta = 1$ (This applies even if one would calculate *distances* with another value of η since the volume element is a quantity related to the global geometry of the universe—alternatively, one can use elliptic integrals, as in Sect. B.3 and Feige (1992).) If one has an expression for $\sigma(z)$, then, since

$$dD^P = R_0 \frac{d\sigma}{\sqrt{1 - k\sigma^2}}, \quad \text{(C3)}$$

which follows directly from Eq. (1), one can write

$$dV(\sigma) = 4\pi R_0^3 \int_0^\sigma \frac{\sigma'^2 d\sigma'}{\sqrt{1 - k\sigma'^2}}. \quad \text{(C4)}$$

where R_0 is given by Eq. (7) for the present values:

$$R_0 = \frac{c}{H_0} \frac{1}{\sqrt{|\Omega_0 + \lambda_0 - 1|}}. \quad \text{(C5)}$$

and

$$\sigma = \frac{D_{y0}(1 + z_y)}{R_0} \quad \text{(C6)}$$

Integration gives

$$V(\sigma) = \begin{cases} 2\pi r^3 \left(\frac{\sqrt{1+\sigma^2}}{\sigma^2} - \frac{arcsinh\,\sigma}{\sigma^3}\right) & k = -1 \\ \frac{4}{3}\pi r^3 & k = 0 \\ 2\pi r^3 \left(\frac{\arcsin\sigma}{\sigma^3} - \frac{\sqrt{1-\sigma^2}}{\sigma^2}\right) & k = +1 \end{cases} \quad \text{(C7)}$$

Thus, for $k = +1$, the total volume of the universe is $2\pi^2 R_0^3$. (See, e.g., Sandage (1995), Sect. 1.6.1; Sandage's d is our D^P and his r is our σ.) Since

$$d\chi = \frac{d\sigma}{\sqrt{1 - k\sigma^2}}$$

Eq. (C7) can also be written as (cf. Feige (1992), Eq. (116); Feige's r is our σ)

$$V(\chi) = 2\pi R_0^3 \begin{cases} \sinh(\chi)\cosh(\chi) - \chi & k = -1 \\ \frac{2}{3}\chi^3 & k = 0 \\ \chi - \sin(\chi)\cos(\chi) & k = +1 \end{cases} \quad \text{(C8)}$$

Of course, all this refers to volumes *now* at the distance corresponding to $z = y$. If the volume at another time is important, say at the time of emission of the light we see now—for instance if one is concerned with the space density of some comoving objects—then the volume element must be divided by $(1 + z)^3$.

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