Appendix to 'Variational Principles for slender fluxtubes I' (A. Achterberg)

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A. Pressure perturbations in a stratified, rotating flow

In this Appendix I briefly derive the equation describing the small pressure perturbations which are induced by the motion of a flux tube in a (differentally) rotating, stratified flow. I will assume that the unperturbed fluid is rotating on cylinders

$$\mathbf{V}_{\rm e} = \Omega(R)\, R\,\hat{\boldsymbol{\theta}}\,, \tag{A1}$$

and is cylindrically symmetric ($\partial/\partial\theta = 0$). I will employ cylindrical coordinates R, θ and Z. Unperturbed quantities will be denoted with a subscript 0.

The equation of motion describing the unperturbed flow reads in this case

$$\begin{aligned} \frac{1}{\rho_{\rm e}}\frac{\partial P_{\rm e}}{\partial R} &= \Omega^2(R)\,R - \frac{\partial\Phi}{\partial R}\,, \\ \frac{1}{\rho_{\rm e}}\frac{\partial P_{\rm e}}{\partial Z} &= -\frac{\partial\Phi}{\partial Z}\,. \end{aligned} \tag{A2}$$

Here $\Phi(R,Z)$ is the gravitational potential. I will neglect the effect of self-gravity of the material, assuming that Φ is given. The assumption that the fluid is rotating on cylinders implies that it is barotropic:

$$\boldsymbol{\nabla} P_{\rm e} \times \boldsymbol{\nabla}\rho_{\rm e} = 0\,. \tag{A3}$$

Consider small perturbations, which displace fluid elements according to $\mathbf{x} \Longrightarrow \mathbf{x}+\boldsymbol{\xi}$. Small perturbations $\delta\mathbf{V}$ in velocity are given in terms of the Lagrangian displacement vector $\boldsymbol{\xi}$ as (e.g. Frieman & Rotenberg, 1960; Lynden-Bell & Ostriker, 1967)

$$\delta\mathbf{V} = D_0\,\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\mathbf{V}\,. \tag{A4}$$

Here D_0 is the comoving time derivative of the unperturbed flow, in this case given by

$$D_0 \equiv \frac{\partial}{\partial t} + \Omega(R)\,\frac{\partial}{\partial\theta}\,. \tag{A5}$$

The linearized equation of motion reads in component form (compare for instance Shu, 1992, p. 137, or Chandrasekhar, 1961, Ch. VII):

$$\begin{aligned} D_0\delta V_{\rm R} - 2\Omega\,\delta V_\theta &= -\frac{1}{\rho_{\rm e}}\frac{\partial P_{\rm r}}{\partial R} + \left(\frac{\delta\rho}{\rho_{\rm e}}\right)\left(\frac{1}{\rho_{\rm e}}\frac{\partial P_{\rm e}}{\partial R}\right)\,, \\ D_0\delta V_\theta + \frac{\kappa^2}{2\Omega}\,\delta V_{\rm R} &= -\frac{1}{\rho_{\rm e} R}\frac{\partial P_{\rm r}}{\partial\theta}\,, \\ D_0\delta V_{\rm Z} &= -\frac{1}{\rho_{\rm e}}\frac{\partial P_{\rm r}}{\partial Z} + \left(\frac{\delta\rho}{\rho_{\rm e}}\right)\left(\frac{1}{\rho_{\rm e}}\frac{\partial P_{\rm e}}{\partial Z}\right)\,. \end{aligned} \tag{A6}$$

Here κ^2 is the epicyclic frequency defined in Eqn. (61) of the main paper.

The Eulerian pressure variation $\delta P \equiv P_{\rm r}$ and density variation $\delta\rho$ are, for isentropic disturbances, given by

$$\begin{aligned} P_{\rm r} &= -\gamma P_{\rm e}\,(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})P_{\rm e}\,, \\ \delta\rho &= -\rho_{\rm e}\,(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\rho_{\rm e}\,. \end{aligned} \tag{A7}$$

In the case of the pressure induced by the motion of a small object through the flow one can introduce a major simplification: compressibility effects can be neglected to lowest order. This corresponds to the Boussinesq approximation, where one puts

$$(\boldsymbol{\nabla}\cdot\delta\mathbf{V}) \approx 0 \quad , \quad (\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) \approx 0\,. \tag{A8}$$

The second equation follows from the first and relation (**??**). Formally, this corresponds to taking the limit of infinite sound speed: $s_{\rm e}^2 \equiv \gamma P_{\rm e}/\rho_{\rm e} \to \infty$. By eliminating $(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})$ between the two equations (**??**) one finds

$$\frac{\delta\rho}{\rho_{\rm e}} \approx \frac{1}{\gamma}\,(\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\left[\ln\left(P_{\rm e}\rho_{\rm e}^{-\gamma}\right)\right] \tag{A9}$$

where, again in the Boussinesq limit, I neglected a term $P_{\rm r}/\gamma P_{\rm e} = P_{\rm r}/(\rho_{\rm e} s_{\rm e}^2)$.

Taking the time derivative D_0 of the R and Z components of the equation of motion, and substituting (**??**) in the resulting expressions one finds the following set of equations:

$$\begin{aligned} D_0^2\,\delta V_{\rm R} - 2\Omega\,D_0\,\delta V_\theta + N^2_{\rm RR}\,\delta V_{\rm R} \;+\; N^2_{\rm RZ}\,\delta V_{\rm Z} &= -\frac{D_0}{\rho_{\rm e}}\left[\frac{\partial P_{\rm r}}{\partial R}\right]\,, \\ D_0^2\,\delta V_{\rm Z} + N^2_{\rm ZZ}\,\delta V_{\rm Z} \;+\; N^2_{\rm ZR}\,\delta V_{\rm R} &= -\frac{1}{\rho_{\rm e}}\left[\frac{\partial(D_0 P_{\rm r})}{\partial Z}\right]\,. \end{aligned} \tag{A10}$$

Here I have defined the tensor $(\mathbf{N}^2)$ (not the square of $\mathbf{N}$!) which has the components

$$\left(N^2\right)_{ij} = -\left[\frac{1}{\gamma\rho_{\rm e}}\frac{\partial P_{\rm e}}{\partial x^i}\right]\frac{\partial}{\partial x^j}\left[\ln\left(P_{\rm e}\rho_{\rm e}^{-\gamma}\right)\right]\,. \tag{A11}$$

The Brunt-Väisälä frequency is defined as:

$$\begin{aligned} N^2 &\equiv -\left[\frac{1}{\gamma\rho_{\rm e}}\,\boldsymbol{\nabla} P_{\rm e}\right]\cdot\boldsymbol{\nabla}\left[\ln\left(P_{\rm e}\rho_{\rm e}^{-\gamma}\right)\right] \\ &= N^2_{\rm RR} + N^2_{ZZ}\,, \end{aligned} \tag{A12}$$

which motivates the notation employed here. The barotropic condition (**??**) implies $N^2_{RZ} = N^2_{ZR}$, and $N^2_{RR}N^2_{ZZ} = \left(N^2_{RZ}\right)^2$.

For notational purposes I introduce the acceleration vector due to the pressure perturbation:

$$\mathbf{a} = -\frac{1}{\rho_{\rm e}}\,\boldsymbol{\nabla} P_{\rm r}\ . \tag{A13}$$

By eliminating $D_0\ \delta V_\theta$ from these equations using (??) one finds, after some straightforward algebra, the following wave equation which links the velocity perturbation to the pressure force:

$$\left(D_0^4 + \left[\kappa^2 + N^2\right] D_0^2 + \kappa^2 N^2_{\rm ZZ}\right)\ \delta\mathbf{V} = \mathbf{Q}\ . \tag{A14}$$

The vector $\mathbf{Q}$ has the components:

$$Q_{\rm R}\ =\ \left(D_0^2 + N^2_{\rm ZZ}\right)\left[D_0 a_{\rm R} + 2\Omega\, a_\theta\right] - N^2_{\rm RZ} D_0 a_{\rm Z}\ ,$$

$$\begin{aligned} Q_\theta\ &=\ \left(D_0^2 + N^2\right) D_0 a_\theta\ - \\ &\quad -\frac{\kappa^2}{2\Omega}\ \left[\left(D_0^2 + N^2_{\rm ZZ}\right) a_{\rm R} - N^2_{\rm RZ} a_{\rm Z}\right]\ , \end{aligned} \tag{A15}$$

$$Q_{\rm Z}\ =\ \left(D_0^2 + \kappa^2 + N^2_{\rm RR}\right) D_0 a_{\rm Z} - N^2_{\rm ZR}\ \left[D_0 a_{\rm R} + 2\Omega\, a_\theta\right]\ .$$

I now employ the slender flux tube approximation, assuming that the radius of the tube is much less than the cylindrical radius, or the pressure and density gradient scale:

$$a \ll \min\left(R,\ L_{\rm P} \equiv \|\boldsymbol{\nabla} \ln P_{\rm e}\|^{-1},\ L_\rho \equiv \|\boldsymbol{\nabla} \ln \rho_{\rm e}\|^{-1}\right)\ . \tag{A16}$$

This means that the induced pressure $P_{\rm r}$ and the associated velocity perturbation $\delta\mathbf{V}$ will generally vary on a small scale $\sim a$. In the Boussinesq approximation this means that one can then neglect gradients in $\rho_{\rm e}$, Ω, κ and N^2_{ij} with respect to gradients in $\delta\mathbf{V}$ and $P_{\rm r}$ *at this stage of the calculation.* The leading order effects of the stratification are already taken into account by the appearance of the terms $\propto N^2_{ij}$.

Taking the divergence of Eqn. (??) yields because of the Boussinesq approximation (??):

$$\boldsymbol{\nabla}\cdot\mathbf{Q} \approx 0. \tag{A17}$$

This equation, together with the 'impenetrable boundary' condition

$$\left(\delta\mathbf{V} - \mathbf{V}_{\rm t}\right)\ \cdot \mathbf{n}_{\rm t} = 0 \tag{A18}$$

at the surface (with normal $\mathbf{n}_{\rm t}$) of the tube moving with velocity $\mathbf{V}_{\rm t}$, determines the induced pressure.

I will consider a number of limiting cases. Let a section of the tube be located at a cylindrical radius $R = R_0$, $\theta = \theta_0$ and $Z = Z_0$. Introducing local cartesian coordinates x, y and z, corotating with the local angular frequency $\Omega_0 = \Omega(R_0)$.

$$x = R - R_0,\ y = R_0(\theta - \theta_0),\ z = Z - Z_0\ , \tag{A19}$$

one can approximate $\boldsymbol{\nabla}\cdot\mathbf{Q}$ by its cartesian version, and (??) becomes

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} = 0\ . \tag{A20}$$

The first case is that of a non-rotating fluid ($\Omega = \kappa = 0$), stratified in the $Z-$direction so that $N^2_{RR} = N^2_{RZ} = 0$. Using (??) and (??) one then finds:

$$\boldsymbol{\nabla}^2_\perp\ \left(\frac{\partial^2 P_{\rm r}}{\partial t^2} + N^2 P_{\rm r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 P_{\rm r}}{\partial t^2}\right) = 0. \tag{A21}$$

Here I have defined the Laplace operator in the $x-y$ plane:

$$\boldsymbol{\nabla}^2_\perp \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\ . \tag{A22}$$

Another simple case is that of a radially stratified ($N^2_{RZ} = N^2_{ZZ} = 0$), rotating fluid. In that case the equation for the induced pressure reads

$$\begin{aligned} &\left[\boldsymbol{\nabla}^2_\perp\ \frac{\partial}{\partial t}\ -\ \left(R\,\frac{{\rm d}\Omega}{{\rm d}R}\right)\ \frac{\partial^2}{\partial x \partial y}\right]\left(\frac{\partial P_{\rm r}}{\partial t}\right) + \\ &\qquad +\ \frac{\partial^2}{\partial z^2}\ \left(\frac{\partial^2 P_{\rm r}}{\partial t^2} + \left[\kappa^2 + N^2\right]\ P_{\rm r}\right) = 0\ . \end{aligned} \tag{A23}$$

Here I have written simply $\partial/\partial t$ for the time derivative D_0 in the corotating frame, and used $N^2 = N^2_{RR}$.

Finally, in a rotating fluid stratified in the $Z-$direction, the induced pressure satisfies

$$\begin{aligned} &\left[\boldsymbol{\nabla}^2_\perp\ \frac{\partial}{\partial t}\ -\ \left(R\,\frac{{\rm d}\Omega}{{\rm d}R}\right)\ \frac{\partial^2}{\partial x \partial y}\right]\left(\frac{\partial^2 P_{\rm r}}{\partial t^2} + N^2 P_{\rm r}\right) + \\ &\qquad +\ \frac{\partial^2}{\partial z^2}\left(\frac{\partial^2}{\partial t^2} + \kappa^2\right)\left(\frac{\partial P_{\rm r}}{\partial t}\right) = 0\ . \end{aligned} \tag{A24}$$

The last two equations simplify considerably in the case of uniform rotation (${\rm d}\Omega/{\rm d}R = 0$), or when the tube axis is oriented along the radial ($x-$) or azimuthal ($y-$)direction so that, in the slender flux tube approximation, one can put $\partial P_{\rm r}/\partial x \approx 0$ or $\partial P_{\rm r}/\partial y \approx 0$ respectively.

In the case where the tube motion is harmonic with frequency ω so that $P_{\rm r} \propto e^{-i\omega t}$ one sees that there are resonances at the two natural frequencies of the external medium: the epicyclic frequency κ and the Brunt Väisälä frequency N. In a uniform, non-rotating medium these equations all reduce to the 'simple' case where the pressure satisfies Laplace's equation $\boldsymbol{\nabla}^2 P_{\rm r} = 0$, the case considered in some detail in the main paper.

The equations derived in this Section for the pressure response are closely related to the occurence of low-frequency internal waves. Consider the simple case of uniform rotation where ${\rm d}\Omega/{\rm d}R = 0$ and $\kappa = 2\Omega$. Assuming a pressure perturbation varying in time as $P_{\rm r} \propto e^{-i\omega t}$ in the corotating frame, the equation for the pressure in the local approximation is

$$(\omega^2 - N^2_{\rm ZZ})\ \frac{\partial^2 P_{\rm r}}{\partial x^2} + (\omega^2 - N^2)\ \frac{\partial^2 P_{\rm r}}{\partial y^2} + \qquad \text{(A25)}$$

$$+ 2N^2_{\rm RZ}\ \frac{\partial^2 P_{\rm r}}{\partial x \partial z} + (\omega^2 - [\kappa^2 + N^2_{\rm RR}])\ \frac{\partial^2 P_{\rm r}}{\partial z^2} = 0\ .$$

Assuming plane waves, varying in space as

$$P_{\rm r} \propto e^{ik_{\rm x}x + ik_{\rm y}y + ik_{\rm z}z}\ , \qquad \text{(A26)}$$

this equation yields a dispersion relation which can be written in the form:

$$\omega^2 - \frac{k_{\rm y}^2}{k^2}\, N^2 - \frac{k_{\rm z}^2}{k^2} \left[\kappa^2 + \left(\frac{k_{\rm x}}{k_{\rm z}}\, N^2_{\rm ZZ} - N^2_{\rm RR} \right)^2 \right] = 0\ . \text{(A27)}$$

In the case of axisymmetric waves ($k_{\rm y} = 0$) this equation is also valid when $d\Omega/dR \neq 0$. In deriving this relation I have used $\left(N^2_{\rm RZ}\right)^2 = N^2_{\rm RR}N^2_{\rm ZZ}$. Note that the wave-vector in the $y-$direction is formally given by $k_{\rm y} = m/R$, with m an integer. In a number of limiting cases this dispersion relation reduces to that of known internal (inertial) waves (e.g. Priest, 1982, Ch. 4). In a homogeneous medium ($N^2_{ij} = 0$) this corresponds to the dispersion relation for inertial waves

$$\omega = \pm \frac{k_{\rm z}}{k}\, \kappa = \pm 2\, \frac{\mathbf{k} \cdot \mathbf{\Omega}}{k}\ , \qquad \text{(A28)}$$

where the last relation is valid for uniform rotation. For a non-rotating ($\kappa = 0$) fluid stratified in the $Z-$direction ($N^2 = N^2_{\rm ZZ}, N^2_{\rm RR} = 0$) one finds the dispersion relation for internal gravity waves

$$\omega = \pm N \sin\theta_{\rm g} \qquad \text{(A29)}$$

where $\theta_{\rm g} = \cos^{-1}(k_{\rm z}/k)$.

B. Calculating the reaction force for harmonic motion

In the rest of the Appendix, I will concentrate on the case of the a tube undergoing a harmonic motion with frequency ω in the corotating frame. To simplify the mathematics, I will neglect the cross-terms such as $\partial^2 P_{\rm r}/\partial x \partial y$, assuming for instance uniform rotation, a tube oriented in the $x-$ or $y-$direction, and/or $(N^2)_{ij} \ll N^2$ for $i \neq j$. The general case can in principle be solved by a transformation to principal axes.

Without cross terms, the equation of motion for the medium surrounding the tube can be written in the form

$$\left(\frac{\partial^2}{\partial t^2} + \sigma_i^2 \right) \delta V_i = -\frac{1}{\rho_{\rm e}}\, \frac{\partial^2 P_{\rm r}}{\partial x_i \partial t}\ , \qquad \text{(B1)}$$

where the index i takes the values $i = 1, 2$. Here I have written $\partial/\partial t$ for D_0 and introduced two cartesian coordinates x_1 and x_2 perpendicular to the axis of the tube. For harmonic motion with δV_i, $P_{\rm r} \propto e^{-i\omega t}$ one can solve this equation for the velocity amplitude:

$$\delta V_i = -i\, \frac{\omega}{D_i(\omega)} \left(\frac{1}{\rho_{\rm e}}\, \frac{\partial P_{\rm r}}{\partial x_i} \right)\ . \qquad \text{(B2)}$$

The functions $D_1(\omega)$ and $D_2(\omega)$ are given by

$$D_{1,2}(\omega) = \omega^2 - \sigma^2_{1,2}\ , \qquad \text{(B3)}$$

with σ_1 and σ_2 the natural frequencies of the system. The equation for the induced pressure follows from the Boussinesq condition $\partial\delta V_1/\partial x_1 + \partial\delta V_2/\partial x_2 = 0$ and, in the local approximation where one neglects large-scale gradients in the density, can be written in the form

$$\frac{1}{D_1(\omega)}\, \frac{\partial^2 P_{\rm r}}{\partial x_1^2} + \frac{1}{D_2(\omega)}\, \frac{\partial^2 P_{\rm r}}{\partial x_2^2} = 0\ . \qquad \text{(B4)}$$

For instance, for a tube oriented along the $x-$axis Eqn. (**??**) reduces to this form with $x_1 = y$, $x_2 = z$, $\sigma_1 = 0$ and $\sigma_2 = N$. I will assume that D_1 and D_2 are both positive so that this equation is of the elliptic type.

This equation must be solved subject to two boundary conditions. At the tube surface condition (**??**) applies. Using (**??**) this condition can be seen to correspond to:

$$\left[\frac{\omega\ \cos\theta}{D_1(\omega)} \right] \frac{\partial P_{\rm r}}{\partial x_1} + \left[\frac{\omega\ \sin\theta}{D_2(\omega)} \right] \frac{\partial P_{\rm r}}{\partial x_2} = i\rho_{\rm e} V_\perp\, \cos{(\theta - \beta)}\ . \text{(B5)}$$

Here $V_\perp$ is the magnitude of the velocity-component of the slender tube perpendicular to its axis, and β the position angle of that velocity in the $x_1 - x_2$ plane, i.e. $\mathbf{V}_\perp = V_\perp\ (\cos\beta\ ,\ \sin\beta)$. The angle θ is the polar angle in that plane. The second boundary condition is that the induced pressure vanishes at infinity:

$$P_{\rm r}(x_1,\ x_2) \to 0 \ \text{ when } ||x_1||,\ ||x_2|| \to \infty. \qquad \text{(B6)}$$

By introducing scaled coordinates

$$\tilde{x}_1 = \frac{\omega}{\sqrt{D_2(\omega)}}\, x_1\ \ ,\ \ \tilde{x}_2 = \frac{\omega}{\sqrt{D_1(\omega)}}\, x_1\ , \qquad \text{(B7)}$$

(note the juxtaposition of indices) this equation reduces to the standard form

$$\omega^2\ \left[\frac{\partial^2 P_{\rm r}}{\partial \tilde{x}_1^2} + \frac{\partial^2 P_{\rm r}}{\partial \tilde{x}_2^2} \right] = 0\ . \qquad \text{(B8)}$$

In these coordinates, the surface of a cylindrical tube with radius a corresponds to an ellipse

$$\frac{\tilde{x}_1^2}{a_1^2} + \frac{\tilde{x}_2^2}{a_2^2} = 1\ , \qquad \text{(B9)}$$

with pricipal axes

$$a_1 = \frac{\omega}{\sqrt{D_2(\omega)}}\, a \quad , \quad a_2 = \frac{\omega}{\sqrt{D_1(\omega)}}\, a \ . \tag{B10}$$

It is therefore advantageous to use elliptical cylindrical coordinates $(u,\ v)$ which are defined by (see for instance Morse & Feshbach, 1953, Ch. 5 ; Arfken, 1970, Ch. 2)

$$\tilde{x}_1 = \alpha\ \cosh u\ \cos v \quad , \quad \tilde{x}_2 = \alpha\ \sinh u\ \sin v \ . \tag{B11}$$

In these coordinates Laplace's equation remains separable, and the induced pressure satisfies

$$\frac{\partial^2 P_{\rm r}}{\partial u^2} + \frac{\partial^2 P_{\rm r}}{\partial v^2} = 0 \ . \tag{B12}$$

By choosing

$$\alpha = \sqrt{a_1^2 - a_2^2} = a\ \left[\frac{\omega^2(\sigma_2^2 - \sigma_1^2)}{D_1(\omega) D_2(\omega)}\right]^{1/2} \ , \tag{B13}$$

one immediately sees that the surface of the tube corresponds to the coordinate surface $u = \text{constant} = u_{\rm t}$ with

$$\coth u_{\rm t} = \frac{a_1}{a_2} = \left[\frac{D_1(\omega)}{D_2(\omega)}\right]^{1/2} \tag{B14}$$

Here I have assumed, without loss of generality, that $\sigma_2 > \sigma_1$.

This choice of coordinates also implies that on the surface of the tube (and only there) the angle variable v coincides with the polar angle θ. From (**??**) one has

$$\cot\theta = \frac{x_1}{x_2} = \left[\frac{D_2(\omega)}{D_1(\omega)}\right]^{1/2}\ \coth u\ \cot v \ . \tag{B15}$$

From condition (**??**) it immediately follows that $\cot\theta = \cot v$ on the tube surface $u = u_{\rm t}$.

The boundary conditions (**??**) and (**??**) take the simple form:

$$\left(\frac{\partial P_{\rm r}}{\partial u}\right)_{u=u_{\rm t}} \ = \ i\frac{\sqrt{D_1(\omega)D_2(\omega)}}{\omega}\ \rho_{\rm e} V_\perp a\ \cos\left(v - \beta\right) \ , \tag{B16}$$

$$P_{\rm r}(u\ ,\ v)\ \rightarrow\ 0\ \ \text{when}\ u \rightarrow \infty.$$

Note that $u \approx \ln(2\tilde{r}/\alpha)$ at large distances (so that $\tilde{\varpi} \equiv \sqrt{\tilde{x}_1^2 + \tilde{x}_2^2} \gg a$) from the tube. The solution of Eqn. (**??**) satisfying both boundary conditions reads

$$P_{\rm r} = -i\omega\left(\frac{D_1 D_2}{\omega^4}\right)^{1/2} e^{(u_{\rm t} - u)} \rho_{\rm e} V_\perp a \cos\left(v - \beta\right) \ . \tag{B17}$$

Since $v = \theta$ on the tube surface, the angular dependence of the induced pressure on the tube surface is the same as in the unstratified case discussed in Section 4 of the main paper (see Eqn. (52) of the main paper), where the coordinate system was chosen such that $\beta = 0$). The influence of the stratification and rotation of the external medium, as parametrized by the frequencies σ_1 and σ_2, is given by the factor

$$\left(\frac{D_1(\omega)D_2(\omega)}{\omega^4}\right)^{1/2} = \sqrt{\left(1 - \frac{\sigma_1^2}{\omega^2}\right)\left(1 - \frac{\sigma_2^2}{\omega^2}\right)} \tag{B18}$$

in this expression.

The force on a section of tube can be immediately calculated by integration over the tube surface $u = u_{\rm t},\ v = \theta$ (compare Eqn. (53) of the main paper, with $\mathbf{e}_\varpi = \mathbf{n}_{\rm t}$ and $\partial\mathbf{V}_\perp/\partial t = -i\omega\mathbf{V}_\perp$):

$$\begin{aligned} \delta\mathbf{F} \ &= \ -a\,\delta\ell \int_0^{2\pi} d\theta\ P_{\rm r}(\varpi = a, \theta)\ \mathbf{n}_{\rm t} \\ &= \ (\pi a^2\,\delta\ell\,\rho_0)\ \times i\omega\ \left(\frac{D_1 D_2}{\omega^4}\right)^{1/2}\ \mathbf{V}_\perp \ . \end{aligned} \tag{B19}$$

This force vanishes at the resonant frequencies $\omega = \pm\sigma_1$, $\omega = \pm\sigma_2$ where $D_1(\omega) = 0$ and $D_2(\omega) = 0$ respectively. In the high-frequency limit $||\omega|| \gg \sigma_{1,2}$ one has $D_1 \sim D_2 \sim \omega^2$, and one recovers the result (53), valid for the force on the tube in a homogeneous external medium at rest.

In this derivation I have assumed that D_1 and D_2 are both positive so that $||\omega|| \geq \sigma_2 \geq \sigma_1$). The same result applies in the low-frequency limit as long as D_1 and D_2 have the same sign ($||\omega|| \leq \sigma_1 \leq \sigma_2$), provided one replaces $D_i(\omega)$ by $||D_i(\omega)||$. In the frequency regime $\sigma_1 < ||\omega|| < \sigma_2$ the functions $D_1 > 0$ and $D_2 < 0$ have opposite signs, and the equation for the induced pressure becomes hyperbolic. Using scaled coordinates, with D_2 replaced by $||D_2||$ in Eqn. (**??**), the equation for the induced pressure becomes a wave equation:

$$\frac{\partial^2 P_{\rm r}}{\partial u^2} - \frac{\partial^2 P_{\rm r}}{\partial v^2} = 0 \ . \tag{B20}$$

This equation has no solution where $P_{\rm r}$ decays at infinity, signaling a breakdown in this case of the validity of the approximations used to derive the equation governing the induced pressure.

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